On the local smoothness of weak solutions to the MHD system near the boundary

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1 Introduction

Assume $\Omega \subset \mathbb{R}^3$ is a C^2- smooth bounded domain and $Q_T = \Omega \times (0,T)$. In this paper we investigate the boundary regularity of solutions to the principal system of magnetohydrodynamics (the MHD equations):

$$\left.\begin{array}{c} \partial_t v + (v\cdot\nabla)v - \Delta v + \nabla p = \operatorname{rot} H \times H \\ \operatorname{div} v = 0 \end{array}\right\} \quad \text{in } Q_T, \tag{1}$$

$$\left.\begin{array}{c} \partial_t H + \operatorname{rot}\operatorname{rot} H = \operatorname{rot}(v\times H) \\ \operatorname{div} H = 0 \end{array}\right\} \quad \text{in } Q_T. \tag{2}$$

Here unknowns are the velocity field $v : Q_T \to \mathbb{R}^3$, pressure $p : Q_T \to \mathbb{R}$, and the magnetic field $H : Q_T \to \mathbb{R}^3$. We impose on v and H the boundary conditions:

$$v|_{\partial\Omega\times(0,T)} = 0, \quad H_\nu|_{\partial\Omega\times(0,T)} = 0, \quad (\operatorname{rot} H)_\tau|_{\partial\Omega\times(0,T)} = 0, \tag{3}$$

Here by ν we denote the outer normal to $\partial\Omega$ and $H_\nu = H\cdot\nu$, $(\operatorname{rot} H)_\tau = \operatorname{rot} H - \nu(\operatorname{rot} H\cdot\nu)$. We introduce the following definition:

Definition: Assume $\Gamma \subset \partial\Omega$. The functions (v,H,p) are called a *boundary suitable weak solution* to the system (1), (2) near $\Gamma_T \equiv \Gamma\times(0,T)$ if

1) $v \in L_{2,\infty}(Q_T)\cap W_2^{1,0}(Q_T)\cap W^{2,1}_{\frac{9}{8},\frac{3}{2}}(Q_T)$,

   $H \in L_{2,\infty}(Q_T)\cap W_2^{1,0}(Q_T)$,

2) $p \in L_{\frac{3}{2}}(Q_T)\cap W^{1,0}_{\frac{9}{8},\frac{3}{2}}(Q_T)$,

3) $\operatorname{div} v = 0$, $\operatorname{div} H = 0$ a.e. in Q_T,

4) $v|_{\partial\Omega} = 0$, $H_\nu|_{\partial\Omega} = 0$ in the sense of traces,

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5) for any $w \in L_2(\Omega)$ the functions

$$t \mapsto \int\limits_\Omega v(x,t)\cdot w(x)\ dx \qquad \text{and} \qquad t \mapsto \int\limits_\Omega H(x,t)\cdot w(x)\ dx$$

are continuous,

6) (v,H) satisfy the following integral identities: for any $t\in[0,T]$

$$\int\limits_\Omega v(x,t)\cdot\eta(x,t)\ dx - \int\limits_\Omega v_0(x)\cdot\eta(x,0)\ dx +$$
$$+\int\limits_0^t\int\limits_\Omega \Big(-v\cdot\partial_t\eta + (\nabla v - v\otimes v + H\otimes H):\nabla\eta - (p+\tfrac12|H|^2)\operatorname{div}\eta\Big)\ dxdt \ = \ 0,$$

for all $\eta\in W^{1,1}_{\frac52}(Q_t)$ such that $\eta|_{\partial\Omega\times(0,t)}=0$,

$$\int\limits_\Omega H(x,t)\cdot\psi(x,t)\ dx \ - \int\limits_\Omega H_0(x)\cdot\psi(x,0)\ dx +$$
$$+\int\limits_0^t\int\limits_\Omega \Big(-H\cdot\partial_t\psi + \operatorname{rot}H\cdot\operatorname{rot}\psi - (v\times H)\cdot\operatorname{rot}\psi\Big)\ dxdt \ = 0,$$

for all $\psi\in W^{1,1}_{\frac52}(Q_t)$ such that $\psi_\nu|_{\partial\Omega\times(0,t)}=0$.

7) For every $z_0=(x_0,t_0)\in\Gamma_T$ such that $\Omega_R(x_0)\times(t_0-R^2,t_0)\subset Q_T$ and for any $\zeta\in C_0^\infty(B_R(x_0)\times(t_0-R^2,t_0])$ such that $\frac{\partial\zeta}{\partial\nu}\Big|_{\partial\Omega}=0$ the following "local energy inequality near Γ_T" holds:

$$\begin{aligned}
&\sup_{t\in(t_0-R^2,t_0)}\int\limits_{\Omega_R(x_0)}\zeta\Big(|v|^2+|H|^2\Big)\ dx + \\
&+2\int\limits_{t_0-R^2}^{t_0}\int\limits_{\Omega_R(x_0)}\zeta\Big(|\nabla v|^2+|\operatorname{rot}H|^2\Big)\ dxdt \ \le \\
&\le \int\limits_{t_0-R^2}^{t_0}\int\limits_{\Omega_R(x_0)}\Big(|v|^2+|H|^2\Big)(\partial_t\zeta+\Delta\zeta)\ dxdt + \\
&+\int\limits_{t_0-R^2}^{t_0}\int\limits_{\Omega_R(x_0)}\Big(|v|^2+2\bar p\Big)v\cdot\nabla\zeta\ dxdt + \\
&-2\int\limits_{t_0-R^2}^{t_0}\int\limits_{\Omega_R(x_0)}(H\otimes H):\nabla^2\zeta\ dxdt + \\
&+2\int\limits_{t_0-R^2}^{t_0}\int\limits_{\Omega_R(x_0)}(v\times H)(\nabla\zeta\times H)\ dxdt
\end{aligned} \tag{4}$$

We remark also that the following identity holds

$$\int\limits_{t_0-R^2}^{t_0}\int\limits_{\Omega_R(x_0)}(v\times H)(\nabla\zeta\times H)\ dxdt \ =$$
$$=\int\limits_{t_0-R^2}^{t_0}\int\limits_{\Omega_R(x_0)}(v\cdot\nabla\zeta)|H|^2\ dxdt \ - \int\limits_{t_0-R^2}^{t_0}\int\limits_{\Omega_R(x_0)}(v\cdot H)(H\cdot\nabla\zeta)\ dxdt$$

Here $L_{s,l}(Q_T)$ is the anisotropic Lebesgue space equipped with the norm

$$\|f\|_{L_{s,l}(Q_T)} := \Big(\int\limits_0^T \Big(\int\limits_\Omega |f(x,t)|^s \ dx \Big)^{l/s} dt \Big)^{1/l},$$

and we use the following notation for the functional spaces:

$$W^{1,0}_{s,l}(Q_T) \equiv L_l(0,T;W^1_s(\Omega)) = \{ \ u \in L_{s,l}(Q_T): \ \nabla u \in L_{s,l}(Q_T) \ \},$$
$$W^{2,1}_{s,l}(Q_T) = \{ \ u \in W^{1,0}_{s,l}(Q_T): \ \nabla^2 u, \ \partial_t u \in L_{s,l}(Q_T) \ \},$$
$$\overset{\circ}{W}{}^1_s(\Omega) = \{ \ u \in W^1_s(\Omega): \ u|_{\partial\Omega} = 0 \ \},$$

and the following notation for the norms:

$$\|u\|_{W^{1,0}_{s,l}(Q_T)} = \|u\|_{L_{s,l}(Q_T)} + \|\nabla u\|_{L_{s,l}(Q_T)},$$
$$\|u\|_{W^{2,1}_{s,l}(Q_T)} = \|u\|_{W^{1,0}_{s,l}(Q_T)} + \|\nabla^2 u\|_{L_{s,l}(Q_T)} + \|\partial_t u\|_{L_{s,l}(Q_T)},$$

Denote $B(x_0,r)$ the open ball in $\mathbb{R}^3$ of radius r centered at x_0 and denote by $B^+(x_0,r)$ the half–ball $\{x \in B(x_0,r) \mid x_3 > 0\}$. For $z_0 = (x_0,t_0)$ denote $Q(z_0,r) = B(x_0,r) \times (t_0 - r^2, t_0)$, $Q^+(z_0,r) = B^+(x_0,r) \times (t_0 - r^2, t_0)$. In this paper we shall use the abbreviations: $B(r) = B(0,r)$, $B^+(r) = B^+(0,r)$ etc, $B = B(1)$, $B^+ = B^+(1)$ etc.

2 Main Results

Our work deals with the sufficient conditions of local regularity of suitable weak solutions to the MHD system near the plane part of the boundary. In [9] the following results were obtained.

Theorem 2.1. *There exists an absolute constant $\varepsilon_* > 0$ with the following property. Assume (v,H,p) is a boundary suitable weak solution in Q_T and assume $z_0 = (x_0,t_0) \in \partial\Omega \times (0,T)$ is such that x_0 belongs to the plane part of $\partial\Omega$. If there exists $r_0 > 0$ such that $Q^+(z_0,r_0) \subset Q_T$ and*

$$\frac{1}{r_0^2} \int\limits_{Q^+(z_0,r_0)} \Big(\ |v|^3 + |H|^3 + |p|^{\frac32} \ \Big) dxdt \ < \ \varepsilon_*,$$

then the functions v and H are Hölder continuous on $\bar{Q}^+(z_0, \frac{r_0}{2})$.

Theorem 2.2. *For any $K > 0$ there exists $\varepsilon_0(K) > 0$ with the following property. Assume (v,H,p) is a boundary suitable weak solution in Q_T and assume $z_0 = (x_0,t_0) \in \partial\Omega \times (0,T)$ is such that x_0 belongs to the plane part of $\partial\Omega$. If*

$$\limsup_{r\to 0} \Big(\ \frac{1}{r} \int\limits_{Q(z_0,r)} |\nabla H|^2 \ dxdt \ \Big)^{1/2} \ < \ K \tag{5}$$

and

$$\limsup_{r\to 0}\Big(\ \frac{1}{r}\int\limits_{Q(z_0,r)}|\nabla v|^2\ dxdt\ \Big)^{1/2}\ <\ \varepsilon_0, \tag{6}$$

then there exists $\rho_* > 0$ *such that the functions* v *and* H *are Hölder continuous on the closure of* $Q^+(z_0,\rho_*)$.

Let us explain the main differences between these theorems. The statement of the theorem 2.1 contains smallness conditions on the three functionals, but these conditions have to hold only for one value of cylinder radius. In theorem 2.2 we have conditions for all sufficiently small values of radius, but smallness condition (6) is imposed only on the velocity v.

To describe more conditions of local regularity we will need the following notations

$$\begin{aligned}
E(r)\ &=\ \Big(\ \tfrac{1}{r}\int\limits_{Q^+(r)}|\nabla v|^2\ dxdt\ \Big)^{1/2},\\
E_*(r)\ &=\ \Big(\tfrac{1}{r}\int\limits_{Q^+(r)}|\nabla H|^2\ dxdt\Big)^{1/2},\\
A(r) &\equiv \Big(\tfrac{1}{r}\sup_{t\in(-r^2,0)}\int\limits_{B^+(r)}|v|^2\ dy\Big)^{1/2},\\
A_*(r) &\equiv \Big(\tfrac{1}{r}\sup_{t\in(-r^2,0)}\int\limits_{B^+(r)}|H|^2\ dy\Big)^{1/2},\\
C_q(r) &\equiv \Big(\tfrac{1}{r^{5-q}}\int\limits_{Q^+(r)}|v|^q\ dydt\Big)^{1/q},\\
F_q(r)\ &=\ \Big(\tfrac{1}{r^{5-q}}\int\limits_{Q^+(r)}|H|^q\ dxdt\Big)^{1/q}\\
D(r) &\equiv \Big(\tfrac{1}{r^2}\int\limits_{Q^+(r)}|p-[p]_{B^+(r)}|^{3/2}\ dydt\Big)^{2/3},\\
D_s(r) &= R^{\frac{5}{3}-\frac{3}{s}}\Big(\int\limits_{-r^2}^{0}\Big(\int\limits_{B^+(r)}|\nabla p|^s\ dy\Big)^{\frac{1}{s}\cdot\frac{3}{2}}\ dt\Big)^{2/3},\\
C(r) = C_3(r),\qquad & F(r) = F_3(r),\qquad D_*(r) = D_{\frac{36}{35}}(r).
\end{aligned} \tag{7}$$

Note that the equations (1), (2), as well as the functionals (7) and the statements of the previous theorems are invariant under the scaling transformations

$$\left.\begin{aligned}
v_\rho(y,s) &= \rho v(\rho y + x_0, \rho^2 s + t_0),\\
H_\rho(y,s) &= \rho H(\rho y + x_0, \rho^2 s + t_0),\\
p_\rho(y,s) &= \rho^2 p(\rho y + x_0, \rho^2 s + t_0).
\end{aligned}\right\} \tag{8}$$

We use the approach which was originally developed in [4] for the Navier-Stokes equations (and later it was used also in [2]). According to this approach the regularity of solutions follows if one of the functionals (7) is bounded uniformly with respect to r and additionally one of these functionals is small only

for a single sufficiently small value of the radius. Our goal is to obtain the same result for the solutions to the MHD system.

The main result of our work is the following theorem, that is a kind of "interpolation" of theorems 2.1 and 2.2.

Theorem 2.3. *For arbitrary $K > 0$ there is a constant $\varepsilon_1(K) > 0$ with the following property. Assumw (v, H, p) is a suitable weak solution to the MHD system in Q_T and $z_0 = (x_0, t_0) \in \partial\Omega \times (0, T)$ where x_0 belongs to the plane part of $\partial\Omega$. If*

$$\limsup_{r\to 0} \Big(\frac{1}{r^2} \int\limits_{Q^+(z_0,r)} |v|^3 \, dxdt \Big)^{1/3} + \Big(\frac{1}{r^3} \int\limits_{Q^+(z_0,r)} |H|^2 \, dxdt \Big)^{1/2} \; < \; K \quad (9)$$

and one of the following conditions holds

$$\begin{gathered} \liminf_{r\to 0} \Big(\frac{1}{r} \int\limits_{Q^+(z_0,r)} |\nabla v|^2 \, dxdt \Big)^{1/2} < \; \varepsilon_1, \\ \liminf_{r\to 0} \Big(\frac{1}{r} \sup_{-r^2<t<0} \int\limits_{B^+(x_0,r)} |v|^2 \, dxdt \Big)^{1/2} \; < \; \varepsilon_1, \\ \liminf_{r\to 0} \Big(\frac{1}{r^2} \int\limits_{Q^+(z_0,r)} |v|^3 \, dxdt \Big)^{1/3} < \; \varepsilon_1, \end{gathered} \quad (10)$$

then there exists $\rho_* > 0$ such that the functions v and H are Hölder continuous on the closure of $Q^+(z_0, \rho_*)$.

Note that it is possible to prove a lot of analogues of theorem 2.3. Generally the proof consist of two steps. The first step is the proof of boundedness of energy functionals (7). Usually to do this it is sufficient to have boundedness condition for one functional depending on v and for another one depending on H. In our work this step is carried out in Section 4. Also we will use some estimates for the magnetic field H, that can be obtained if we consider equation (2) as the heat equation with lower order terms depending on v. This inequalities are proved in Section 3.

The second step is the proof of regularity condition when all of functionals (7) are bounded and one of functionals on v is small for a single sufficiently small value of r. This result can be found in Section 5.

3 Estimates of Solutions to the Heat Equation

In this section we study solutions of the heat equations with the lower order terms:

$$\begin{gathered} \partial_t H \; - \; \Delta H \; = \; \operatorname{div}(v \otimes H - H \otimes v) \qquad \text{in} \quad Q^+. \\ v|_{x_3=0} = 0, \\ H_3|_{x_3=0} = 0, \qquad H_{\alpha,3}|_{x_3=0} = 0, \qquad \alpha = 1, 2. \end{gathered}$$

Namely, we assume the functions (v,H) possess the following properties:

$$\begin{gathered} v,\ H\in W^{1,0}_2(Q^+),\\ v|_{x_3=0}=0,\quad H_3|_{x_3=0}=0\quad\text{in the sense of traces,}\end{gathered}\tag{11}$$

for any $\eta\in C_0^\infty(Q;\mathbb{R}^3)$ such that $\eta_3|_{x_3=0}=0$ the following integral identity holds

$$\int\limits_{Q^+}\Big(-H\cdot\partial_t\eta+\nabla H:\nabla\eta\Big)\,dxdt\ =\ -\int\limits_{Q^+}G:\nabla\eta\ dxdt,\tag{12}$$

here $G\ =\ v\otimes H-H\otimes v$, and

$$\operatorname{div}v=0,\qquad\operatorname{div}H=0\qquad\text{a.e. in}\quad Q^+.\tag{13}$$

Lemma 3.1. *Assume that conditions (11) — (13) hold. Then for any* $0<r\le 1$ *and* $0<\theta\le 1$ *the following estimate holds*

$$F_2(\theta r)\le c\theta^\alpha F_2(r)+c\theta^{-\frac32}C(r)A_*(r).\tag{14}$$

Proof. Denote by v^* and H^* the extensions of functions v and H from Q^+ onto Q. Fix arbitrary $r\in(0,1)$ and let $\zeta\in C^\infty(\bar Q)$ be a cut off function such that $\zeta\equiv1$ on $Q(r)$ and $\operatorname{supp}\zeta\subset B\times(-1,0]$. Denote $\Pi=\mathbb{R}^3\times(-1,0)$ and denote by $\hat G$ the function which coincides with G^* on $Q(\frac r2)$ and additionally possesses the following properties: $\hat G\in W^{1,0}_1(\Pi)\cap L_{\frac{18}{11},\frac65}(\Pi)$, $\hat G$ is compactly supported in Π, and

$$\|\hat G\|_{L_{\frac65,2}(\Pi)}\ \le\ c\|G^*\|_{L_{\frac65,2}(Q(\frac r2))}\ \le\ c\|G\|_{L_{\frac65,2}(Q^+(\frac r2))}\tag{15}$$

We decompose H^* as

$$H^*\ =\hat H\ +\tilde H,$$

where $\hat H$ is a solution of the Cauchy problem for the heat equation

$$\left\{\begin{array}{l}\partial_t\hat H-\Delta\hat H\ =\ \operatorname{div}\hat G\qquad\text{in}\quad\Pi,\\ \hat H|_{t=-1}=0,\end{array}\right.\tag{16}$$

defined by the formula $\hat H=\Gamma*\operatorname{div}\hat G=-\nabla\Gamma*\hat G$, where Γ is the fundamental solution of the heat operator. The function $\tilde H$ satisfies the homogeneous heat equation

$$\partial_t\tilde H-\Delta\tilde H=0\qquad\text{in}\quad Q(\tfrac r2).\tag{17}$$

Take arbitrary $\theta\in(0,\frac12)$. We estimate $\|H\|_{L_2(Q^+(\theta r))}$ in the following way

$$\|H\|_{L_2(Q^+(\theta r))}\ \le\ \|H^*\|_{L_2(Q(\theta r))}\ \le\ \|\hat H\|_{L_2(Q(\theta r))}\ +\ \|\tilde H\|_{L_2(Q(\theta r))},\tag{18}$$

For $\|\hat H\|_{L_2(Q(\theta r))}$ we have

$$\|\hat H\|_{L_2(Q(\theta r))}\ \le\ c\ \|\hat H\|_{L_2(Q(\frac r2))}.\tag{19}$$

As $\tilde{H}$ satisfies (17) by local estimate of the maximum of $\tilde{H}$ via its L_2–norm we obtain

$$\begin{aligned} \|\tilde{H}\|_{L_2(Q(\theta r))} \ &\le\ c\, \theta^{\frac{5}{2}}\ \|\tilde{H}\|_{L_2(Q(\frac{r}{2}))}\ \le \\ &\le\ c\, \theta^{\frac{5}{2}}\ (\|H^*\|_{L_2(Q(r))} + \|\hat{H}\|_{L_2(Q(\frac{r}{2}))}) \end{aligned} \tag{20}$$

So, we need to estimate $\|\hat{H}\|_{L_2(Q(\frac{r}{2}))}$. As singular integrals are bounded on the anisotropic Lesbegue space $L_{s,l}$ (see, for example, [8]) for the convolution $\hat{h} = \Gamma * \hat{G}$ we obtain the estimate

$$\|\hat{h}\|_{W^{2,1}_{\frac{6}{5},2}(Q(r))}\ \le\ c\|\hat{G}\|_{L_{\frac{6}{5},2}(\Pi)}.$$

On the other hand, from the 3D– parabolic imbedding theorem (see [1])

$$W^{2,1}_{s,l}(Q) \hookrightarrow W^{1,0}_{p,q}(Q), \quad \text{as} \quad 1 - \left(\frac{3}{s} + \frac{2}{l} - \frac{3}{p} - \frac{2}{q}\right) \ge 0,$$

for $p = q = 2$ and $s = \frac{6}{5}$, $l = 2$ and for $\hat{H} = -\nabla\hat{h}$ we obtain

$$\|\hat{H}\|_{L_2(Q(r))}\ \le\ c\ \|\hat{G}\|_{L_{\frac{6}{5},2}(\Pi)}.$$

(Note that the constant c in this inequality does not depend on r). Taking into account (15) we arrive at

$$\|\hat{H}\|_{L_2(Q(r))}\ \le\ c\ \|G\|_{L_{\frac{6}{5},2}(Q^+(\frac{r}{2}))}. \tag{21}$$

From the definition of G we obtain

$$\begin{aligned} \|G\|_{L_{\frac{6}{5},2}(Q^+(\frac{r}{2}))}\ &\le\ c\ \left(\int\limits_{-r^2/4}^{0} \|v\otimes H\|^2_{L_{\frac{6}{5}}(B^+(r/2))}\, dt\right)^{\frac{1}{2}} \le \\ &\le \left(\int\limits_{-r^2}^{0} \|v\|^2_{3,B^+(r)} \|H\|^2_{2,B^+(r)} dt\right)^{\frac{1}{2}} \le \\ &\le \|H\|_{2,\infty,Q^+(r)} \left(\int\limits_{-r^2}^{0} \|v\|^2_{3,B^+(r)} dt\right)^{\frac{1}{2}} \le r^{\frac{3}{2}} C(r) A_*(r). \end{aligned} \tag{22}$$

Combining inequalities (20)-(22) we will get the statement of lemma.

□

Using interpolation inequality (24) for $C(r)$ in the right hand side (14), we will obtain inequality (14) in another form

Corollary 3.1. *Assume that conditions (11) — (13) hold. Then for any* $0 < r \le 1$ *and* $0 < \theta \le 1$ *the following estimate holds*

$$F_2(\theta r) \le c\theta^{\alpha} F_2(r) + c\theta^{-\frac{3}{2}} E^{\frac{1}{2}}(r) A^{\frac{1}{2}}(r) A_*(r). \tag{23}$$

4 Boundedness of energy functionals

In this section we derive estimates of energy functionals which allow us to obtain uniform boundedness (with respect to the radius) of all functionals (7) if boundedness of some of them is known.

Observe that one can prove a group of estimates that are the consequences of Hölder inequality, embedding theorem and interpolation inequality.

$$C(r) \ \leq \ A^{\frac12}(r)E^{\frac12}(r), \qquad F(r) \ \leq \ A_*^{\frac12}(r)[E_*^{\frac12}(r)+F_2^{\frac12}(r)] \tag{24}$$

$$D(r) \ \leq \ cD_1(r), \qquad D_1(r) \ \leq \ cD_s(r), \qquad \forall s>1. \tag{25}$$

First of all we will prove the decay estimate for the pressure

Lemma 4.1. *If v,p,H are the suitable weak solution near the boundary to the MHD equations in Q^+. Then for any $0<r\leq 1$ $0<\theta\leq 1$ the following estimate holds*

$$\begin{gathered} D_{\frac{12}{11}}(\theta r) \leq c\theta^{\alpha}\Big(D_{\frac{12}{11}}(r)+E(r)\Big)+ \\ +c(\theta)\Big(E(r)A^{\frac12}(r)C^{\frac12}(r)+E_*(r)A_*^{\frac12}(r)F^{\frac12}(r)\Big). \end{gathered} \tag{26}$$

Proof. To obtain (26) we apply the method developed in [3], [5], see also [6]. Denote $\Pi_r=\mathbb{R}^3_+\times(-r^2,0)$. We fix $r\in(0,1]$ and $\theta\in(0,\frac12)$ and define a function $g:\Pi_r^+\to\mathbb{R}^3$ by the formula

$$g \ = \ \begin{cases} \operatorname{rot} H\times H-(v\cdot\nabla)v, & \text{in } \ Q^+(r), \\ \qquad 0, & \text{in } \ \Pi_r^+\setminus Q^+(r) \end{cases}$$

Then we decompose v and p as

$$v \ = \ \hat v+\tilde v, \qquad p \ = \ \hat p+\tilde p,$$

where $(\hat v,\hat p)$ is a solution of the Stokes initial boundary value problem in a half-space

$$\begin{cases} \partial_t\hat v-\Delta\hat v+\nabla\hat p \ = \ g, \\ \operatorname{div}\hat v=0 \end{cases} \qquad \text{in} \quad \Pi_r^+,$$

$$\hat v|_{t=0}=0, \qquad \hat v|_{x_3=0}=0,$$

and $(\tilde v,\tilde p)$ is a solution of the homogeneous Stokes system in $Q^+(r)$:

$$\begin{cases} \partial_t\tilde v-\Delta\tilde v+\nabla\tilde p \ = \ 0, \\ \operatorname{div}\tilde v=0 \end{cases} \qquad \text{in} \quad Q^+(r),$$

$$\tilde v|_{x_3=0}=0.$$

For $\nabla\hat p$ and $\nabla\tilde p$ the following estimates hold (see [5], see also [7]):

$$\begin{gathered} \|\nabla\hat p\|_{L_{\frac{12}{11},\frac32}(Q^+(r))} \ + \ \frac1r\|\nabla\hat v\|_{L_{\frac{12}{11},\frac32}(Q^+(r))} \ \leq \\ \leq \ c\ \Big(\ \|H\times\operatorname{rot}H\|_{L_{\frac{12}{11},\frac32}(Q^+(r))} \ + \ \|(v\cdot\nabla)v\|_{L_{\frac{12}{11},\frac32}(Q^+(r))} \ \Big), \end{gathered} \tag{27}$$

$$\|\nabla\tilde p\|_{L_{\frac{12}{11},\frac32}(Q^+(\theta r))} \ \le\ c\,\theta^\alpha \left(\frac1r\|\nabla\tilde v\|_{L_{\frac{12}{11},\frac32}(Q^+(r))} \ +\ \|\nabla\tilde p\|_{L_{\frac{12}{11},\frac32}(Q^+(r))}\right). \quad (28)$$

To estimate the right hand side of (27) we will use Hölder and interpolation inequalities

$$\begin{gathered}\|(v\cdot\nabla)v\|_{\frac{12}{11},\frac32} = \\ = \left(\int\limits_{-r^2}^{0}\|(v\cdot\nabla)v\|^{\frac32}_{\frac{12}{11}}dt\right)^{\frac23} \le \left(\int\limits_{-r^2}^{0}\|\nabla v\|_2^{\frac32}\|v\|^{\frac32}_{\frac{12}{5}}dt\right)^{\frac23} \le \\ \le \|\nabla v\|_2\left(\int\limits_{-r^2}^{0}\|v\|^6_{\frac{12}{5}}dt\right)^{\frac16} \le \|\nabla v\|_2\left(\int\limits_{-r^2}^{0}\|v\|_2^3\|v\|_3^3dt\right)^{\frac16} \le \\ \le \|\nabla v\|_2\|v\|^{\frac12}_{2,\infty}\|v\|_3^{\frac12}\end{gathered} \quad (29)$$

Term $\|H\times\operatorname{rot}H\|_{L_{\frac{12}{11},\frac32}(Q^+(r))}$ can be estimated similarly.

The right hand side of (28) can be estimated as follows

$$\begin{gathered}\frac1r\|\nabla\tilde v\|_{L_{\frac{12}{11},\frac32}(Q^+(r))} \ +\ \|\nabla\tilde p\|_{L_{\frac{12}{11},\frac32}(Q^+(r))} \le \\ \le c\Big(\|\nabla v\|_{2,Q^+(r)} + \|\nabla p\|_{L_{\frac{12}{11},\frac32}(Q^+(r))} + \\ + \frac1r\|\nabla\hat v\|_{L_{\frac{12}{11},\frac32}(Q^+(r))} \ +\ \|\nabla\hat p\|_{L_{\frac{12}{11},\frac32}(Q^+(r))}\Big).\end{gathered} \quad (30)$$

Combining inequalities (27)-(30) we will get the statement of lemma.

□

Theorem 4.1. *Let v,p,H are the suitable weak solution near the boundary to the MHD equations in Q^+ and*

$$C(R)+F_2(R)\le M, \quad 0<R\le 1.$$

If we consider the following functional

$$\mathcal{L}(r) = A^2(r)+E^2(r)+A^2_*(r)+E^2_*(r)+D^{\frac{25}{24}}_{\frac{12}{11}}(r),$$

then the following estimate will hold

$$\mathcal{L}(r)\le C(M)(r^\alpha\mathcal{L}(1)+1).$$

Proof. From local energy inequality we obtain

$$\begin{gathered}\mathcal{L}(\theta r)\le c\big(C_2^2(2\theta r)+F_2^2(2\theta r)+C^3(2\theta r)+C(2\theta r)D(2\theta r)+ \\ C^2(2\theta r)F_3(2\theta r)+C(2\theta r)F_3^2(2\theta r)+D^{\frac{25}{24}}_{\frac{12}{11}}(\theta r)\big).\end{gathered}$$

Now we will estimate every term in the right hand side. Our goal is to prove the following estimate

$$\mathcal{L}(\theta r) \leq \frac{1}{2}\mathcal{L}(r) + C(M). \tag{31}$$

Then we can use a standard iteration procedure (see [4]) and obtain the statement of the theorem.

Estimates for the first three terms are obvious. To estimate the 4th term we use Young inequality and inequality (25)

$$C(2\theta r)D(2\theta r) \leq c\left(D_{\frac{12}{11}}^{\frac{25}{24}}(2\theta r) + M^{25}\right).$$

Now we are going to prove estimate for $D_{\frac{12}{11}}(2\theta r)$. To do this, we will use inequalities (24) and (26)

$$\begin{gathered} D_{\frac{12}{11}}(2\theta r) \leq \\ \leq c\theta^{\alpha}\left(D_{\frac{12}{11}}(r) + E(r)\right) + c(\theta)\left(E(r)A^{\frac{1}{2}}(r)C^{\frac{1}{2}}(r) + E_*(r)A_*^{\frac{1}{2}}(r)F_3^{\frac{1}{2}}(r)\right) \leq \\ \leq c\theta^{\alpha}\left(D_{\frac{12}{11}}(r) + E(r)\right) + c(\theta)\left(\mathcal{L}^{\frac{3}{4}}(r)M^{\frac{1}{2}} + \mathcal{L}^{\frac{3}{4}}(r)F_3^{\frac{1}{2}}(r)\right). \end{gathered} \tag{32}$$

We use interpolation inequality to estimate $F_3(r)$

$$\begin{gathered} F_3(r) \leq F_{\frac{10}{3}}^{\frac{5}{6}}(r)F_2^{\frac{1}{6}}(r) \leq c\left(A_*^{\frac{2}{5}}(r)\left(E_*^{\frac{3}{5}}(r) + F_2^{\frac{3}{5}}(r)\right)\right)^{\frac{5}{6}}F_2^{\frac{1}{6}}(r) \leq \\ \leq c\left(\mathcal{L}^{\frac{5}{12}}(r)M^{\frac{1}{6}} + \mathcal{L}^{\frac{1}{6}}M^{\frac{2}{3}}\right). \end{gathered} \tag{33}$$

Now we substitute this to (32)

$$\begin{gathered} D_{\frac{12}{11}}(2\theta r) \leq c\theta^{\alpha}\left(D_{\frac{12}{11}}(r) + E(r)\right) + \\ + c(\theta)\left(\mathcal{L}^{\frac{3}{4}}(r)M^{\frac{1}{2}} + \mathcal{L}^{\frac{23}{24}}(r)M^{\frac{1}{12}} + \mathcal{L}^{\frac{5}{6}}(r)M^{\frac{1}{3}}\right). \end{gathered}$$

As the result we obtain

$$D_{\frac{12}{11}}^{\frac{25}{24}}(2\theta r) \leq c\theta^{\alpha}\mathcal{L}(r) + c(\theta)\left(\mathcal{L}^{\frac{575}{576}}(r)M^{k_1} + \mathcal{L}^{\frac{125}{144}}(r)M^{k_2}\right).$$

Since the right hand side of the last inequality contain $\mathcal{L}(r)$ in the degree smaller then 1, choosing θ sufficiently small and using Young inequality we obtain an estimate (31).

To estimate the last two terms we use (33)

$$C(r)F_3^2(r) \leq c\left(\mathcal{L}^{\frac{5}{6}}(r)M^{\frac{4}{3}} + \mathcal{L}^{\frac{1}{3}}M^{\frac{7}{3}}\right)$$

and Young inequality. The second term can be estimated in the same maner.

As the result we obtain (31). Next by standard iteration procedure we finish the prove of the theorem.

$\square$

5 Proof of main results

As a first step we obtain theorem 2.1 without smallness condition on a pressure.

Lemma 5.1. *For arbitrary $M > 0$ there is $\varepsilon_1(M) > 0$, such that if v, p, H are the suitable weak solution near the boundary to the MHD equations in Q^+,*

$$A(R) + E(R) + A_*(R) + E_*(R) + F_3(R) + D_{\frac{12}{11}}(R) < M, \quad \forall\, 0 < R \le 1 \tag{34}$$

and

$$C(1) + F_3(1) < \varepsilon_1, \tag{35}$$

then the functions v and H are Hölder continuous on $\bar{Q}^+(r_*)$ for some $0 < r_* < 1$.

Proof. Assume that the statement of the lemma is false. Then there are sequences of v_n, p_n, H_n of suitable weak solutions in Q^+, such that

$$C(v_n, 1) + F_3(H_n, 1) = \varepsilon_n \to 0, \text{ as } n \to \infty \tag{36}$$

and 0 is a singular point. Then by theorem 2.1

$$C(v_n, r) + D(p_n, r) + F_3(H_n, r) > \varepsilon_* \tag{37}$$

for all $0 < r < 1$.

On the other hand from (26), (34), (36) and the embedding theorem we have

$$D(p_n, r) \le c D_{\frac{12}{11}}(p_n, r) \le c r^\alpha M + c(r) M^{\frac{3}{2}} \varepsilon_n^{\frac{1}{2}}. \tag{38}$$

So we fix $0 < r \le 1$ and pass to the limit by n in (37) and (38)

$$\varepsilon_* \le \limsup_{n \to \infty} \left(C(v_n, r) + D(p_n, r) + F_3(H_n, r) \right) =$$
$$= \limsup_{n \to \infty} D(p_n, r) \le c r^\alpha M.$$

As the result we obtain, that the inequality

$$\varepsilon_* \le c r^\alpha M$$

must be true for arbitrary $0 < r < 1$. So we have a contradiction.

□

Theorem 5.1. *For arbitrary $M > 0$ there is $\varepsilon_2(M) > 0$, such that if v, p, H are the suitable weak solution near the boundary to the MHD equations in Q^+, satisfying to (34) and one of the following conditions holds*

$$E(1) < \varepsilon_2, \tag{39}$$

$$A(1) < \varepsilon_2, \tag{40}$$

$$C(1) < \varepsilon_2, \tag{41}$$

then the functions v and H are Hölder continuous on $\bar{Q}^+(r_*)$ for some $0 < r_* < 1$.

Proof. The proof of this theorem is similar to the proof of previous lemma. We begin from the case (39). Let v_n, p_n, H_n are the sequences of suitable weak solutions to the MHD system, such that (34) holds,

$$E(v_n, 1) = \varepsilon_n \to 0$$

as $n \to \infty$, and $z_0 = 0$ is a singular point. Then from lemma 5.1 we have

$$C(v_n, r) + F_3(v_n, r) > \varepsilon_1 \tag{42}$$

for arbitrary $0 < r < 1$.

On the other hand

$$C(v_n, r) \le \frac{c}{r^{\frac{2}{3}}} C(v_n, 1) \le \frac{c}{r^{\frac{2}{3}}} A^{\frac{1}{2}}(v_n, 1) E^{\frac{1}{2}}(v_n, 1) \to 0 \tag{43}$$

as $n \to \infty$ and for any fixed $0 < r \le 1$. From (23) we obtain

$$\limsup F_2(H_n, r) \le c r^{\alpha} M. \tag{44}$$

Next we use interpolation inequality

$$F_3(H_n, r) \le F_2^{\frac{1}{6}}(H_n, r) F_{\frac{10}{3}}^{\frac{5}{6}}(H_n, r). \tag{45}$$

To estimate the second factor in the right hand side of (45) we use (24). So from (42)-(45) we obtain

$$\varepsilon_1 \le \limsup_{n \to \infty} \left(C(v_n, r) + F_3(v_n, r) \right) \le c r^{\alpha_1} M^k \quad \forall\, 0 < r \le \frac{1}{2},$$

and, if we choose r sufficiently small, we will have a contradiction.

Observe, that $E(r)$ and $A(r)$ take part in (23) symmetrically, so the proof of this theorem in the case (40) is similar to the previous one. In the case of (41) for obtaining (44) is sufficient to use (14).

$\square$

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